

## ON HOMOMORPHISMS BETWEEN GLOBAL WEYL MODULES

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ABSTRACT. Let  $\mathfrak{g}$  be a simple finite dimensional Lie algebra and let  $A$  be a commutative associative algebra with unity. Global Weyl modules for the generalized loop algebra  $\mathfrak{g} \otimes A$  were defined in [6, 7] for any dominant integral weight  $\lambda$  of  $\mathfrak{g}$  by generators and relations and further studied in [4]. They are expected to play a role similar to that of Verma modules in the study of categories of representations of  $\mathfrak{g} \otimes A$ . One of the fundamental properties of Verma modules is that the space of morphisms between two Verma modules is either zero or one-dimensional and also that any non-zero morphism is injective. The aim of this paper is to establish an analogue of this property for global Weyl modules. This is done under certain restrictions on  $\mathfrak{g}$ ,  $\lambda$  and  $A$ . A crucial tool is the construction of fundamental global Weyl modules in terms of fundamental local Weyl modules given in Section 3.

## INTRODUCTION

In this paper, we continue the study of representations of loop algebras and, more generally, of Lie algebras of the form  $\mathfrak{g} \otimes A$ , where  $\mathfrak{g}$  is a complex finite-dimensional simple Lie algebra and  $A$  is a commutative associative algebra with unity over the complex numbers. To be precise, we are interested in the category  $\mathcal{I}_A$  of  $\mathfrak{g} \otimes A$ -modules which are integrable as  $\mathfrak{g}$ -modules. One motivation for studying this category is that it is closely related to the categories  $\mathcal{F}$  and  $\mathcal{F}_q$  of finite-dimensional representations, respectively, of affine Lie and quantum affine algebras, which has been a subject of considerable interest in recent years. The categories  $\mathcal{I}_A$ ,  $\mathcal{F}$  and  $\mathcal{F}_q$  fail to be semi-simple, and it was proved in [6] that irreducible representations of the quantum affine algebra specialize at  $q = 1$  to reducible indecomposable representations of the loop algebra (obtained by taking  $A = \mathbf{C}[t, t^{-1}]$ ). This phenomenon is analogous to the one observed in modular representation theory, where an irreducible finite-dimensional representation in characteristic zero becomes reducible by passing to characteristic  $p$  and is called a Weyl module.

This analogy motivated the definition of Weyl modules (global and local) for loop algebras in [6]. Their study was pursued for more general rings  $A$  in [4] and [7]. Thus, given any dominant integral weight of the semi-simple Lie algebra  $\mathfrak{g}$  one can define an infinite-dimensional object  $W_A(\lambda)$  of  $\mathcal{I}_A$ , called the *global Weyl module*, via generators and relations. It was shown (see [4] for the most general case) that if  $A$  is finitely generated, then  $W_A(\lambda)$  is a right module for a certain commutative finitely generated associative algebra  $\mathbf{A}_\lambda$ , which is canonically associated with  $A$  and  $\lambda$ . The local Weyl modules are obtained by tensoring the global Weyl module  $W_A(\lambda)$  over  $\mathbf{A}_\lambda$  with simple  $\mathbf{A}_\lambda$ -modules or, equivalently, can be given via generators and relations.

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The local Weyl modules have been useful in understanding the blocks of the category  $\mathcal{F}_A$  of finite-dimensional representations of  $\mathfrak{g} \otimes A$ , and the quantum analogs are used to understand the blocks in  $\mathcal{F}_q$ . One motivation for this paper is to explore the use of the global Weyl modules to further understand the homological properties of the categories  $\mathcal{F}_A$  and  $\mathcal{I}_A$ . The global Weyl modules have nice universal properties, and one expects them to play a role similar to that of the Verma modules  $M(\lambda)$  in the study of the Bernstein-Gelfand-Gelfand category  $\mathcal{O}$  for  $\mathfrak{g}$ . One of the most basic results about Verma modules is that  $\text{Hom}_{\mathfrak{g}}(M(\lambda), M(\mu))$  is of dimension at most one and also that any non-zero map is injective. In this paper we prove the following analogue of this result.

**Theorem 1.** Suppose that  $\mathfrak{g} = \mathfrak{sl}_{r+1}$  or  $\mathfrak{g} = \mathfrak{sp}_{2r}$  and  $A = \mathbb{C}[t]$  or  $\mathbb{C}[t^{\pm 1}]$ . Then

$$\text{Hom}_{\mathfrak{g} \otimes A}(W_A(\mu), W_A(\lambda)) \cong_{\mathbf{A}_\lambda} \begin{cases} 0, & \mu \neq \lambda \\ \mathbf{A}_\lambda, & \mu = \lambda \end{cases}$$

and every non-zero element of  $\text{Hom}_{\mathfrak{g} \otimes A}(W_A(\lambda), W_A(\lambda))$  is injective.

A more general version of this statement is provided in Theorem 3. Note that instead of having dimension one, the Hom-spaces become free modules of rank one over  $\mathbf{A}_\lambda$ . This of course reflects the fact that the “top” weight space of  $M(\lambda)$  is one-dimensional, while for  $W_A(\lambda)$  it is isomorphic to  $\mathbf{A}_\lambda$  as a  $\mathbf{A}_\lambda$ -module. It should be noted that, although the vanishing of Hom for  $\mu \neq \lambda$  definitely fails for  $\mathfrak{g}$  of other types, as shown in the remark in §6.1, we still expect that all non-zero elements of  $\text{Hom}_{\mathfrak{g} \otimes A}(W_A(\mu), W_A(\lambda))$  are injective.

As is natural, one is guided by the representation theory of quantum affine algebras and the phenomena occurring in  $\mathcal{F}_q$ . It is interesting to note that Theorem 3, for instance, fails exactly when the irreducible finite-dimensional module in the quantum case specializes to a reducible module for the loop algebra. We conclude by noting that global Weyl modules are also defined for quantum affine algebras and similar questions can be posed in the quantum case. Some of these have been studied in [1] using crystal bases. However, it should be noted that the results in the quantum case do not specialize to the classical case. The results in the classical case are sometimes different from those in the quantum situation. One of the reasons is that the local fundamental Weyl module is always irreducible in the quantum case, while its classical analogue is often reducible.

We now explain in more detail the organization and results of this paper. In Section 1 we fix the notation and recall the basic facts on global Weyl modules that will be necessary for the sequel. The main results are formulated and discussed in Section 2, together with their relation with the quantum case. Section 3 contains properties of the algebras  $\mathbf{A}_\lambda$  and local Weyl modules which are needed later. One of the principal difficulties in working with global Weyl modules is their abstract definition. However, when  $A$  is the ring of (generalized) Laurent polynomials, it admits a natural bialgebra structure which can be used to reconstruct a *fundamental* global Weyl module  $W_A(\omega_i)$  from a local Weyl module. This explicit realization of fundamental Weyl modules plays a crucial role in proving our main results and occupies Section 4 (Proposition 4.4). It is not hard to deduce from the definition of global Weyl modules that there exists a canonical map from  $W_A(\lambda)$  to a suitable tensor product of global fundamental Weyl modules. Our next result (Theorem 2 and its Corollary 2.2) studies the more general problem of describing the space of  $\mathfrak{g} \otimes A$ -module homomorphisms from  $W_A(\lambda)$

to an arbitrary tensor product of fundamental global Weyl modules. The structure of the fundamental local Weyl modules was given in [4] for all classical simple Lie algebras and for certain nodes of exceptional Lie algebras. This information and the realization of fundamental global Weyl modules provided by Proposition 4.4 allows one to establish Theorem 2.

In Theorem 3, we use Theorem 2 to compute  $\text{Hom}_{\mathfrak{g} \otimes A}(W_A(\lambda), W_A(\mu))$  when  $A$  is the ring of polynomials or Laurent polynomials in one variable and under suitable conditions on  $\mu$ . We are able to do this because one has a lot of information (see [1, 5, 6, 8, 10, 14]) on global Weyl modules in this case. We are also able to use a result of [7] and Theorem 2 of this paper to compute  $\text{Hom}_{\mathfrak{g} \otimes A}(W_A(\lambda), W_A(\mu))$  when  $\mathfrak{g}$  is of type  $\mathfrak{sl}_{r+1}$ ,  $\mu$  is a multiple of the first fundamental weight and  $A$  is the (generalized) ring of Laurent polynomials in several variables. This is done in Section 6 of the paper.

## 1. GLOBAL WEYL MODULES

In this section we establish the notation to be used in the rest of the paper and then recall the definition and some elementary properties of the global Weyl modules.

**1.1.** Let  $\mathbf{C}$  be the field of complex numbers and let  $\mathbf{Z}$  (respectively  $\mathbf{Z}_+$ ) be the set of integers (respectively non-negative integers). Given two complex vector spaces  $V, W$  let  $V \otimes W$  (respectively,  $\text{Hom}(V, W)$ ) denote their tensor product over  $\mathbf{C}$  and (respectively the space of  $\mathbf{C}$ -linear maps from  $V$  to  $W$ ).

Given a commutative and associative algebra  $A$  over  $\mathbf{C}$ , let  $\text{Max } A$  be the maximal spectrum of  $A$  and  $\text{mod } A$  the category of left  $A$ -modules. Given a right  $A$ -module  $M$  and an element  $m \in M$ , the annihilating (right) ideal of  $m$  is

$$\text{Ann}_A m = \{a \in A : m.a = 0\}.$$

**1.2.** For a complex Lie algebra  $\mathfrak{a}$  let  $\mathbf{U}(\mathfrak{a})$  be the associated universal enveloping algebra. It is that the assignment  $x \mapsto x \otimes 1 + 1 \otimes x$  for  $x \in \mathfrak{a}$  extends to a homomorphism of algebras  $\Delta : \mathbf{U}(\mathfrak{a}) \rightarrow \mathbf{U}(\mathfrak{a}) \otimes \mathbf{U}(\mathfrak{a})$  and defines a bialgebra structure on  $\mathbf{U}(\mathfrak{a})$ . In particular, if  $V, W$  are two  $\mathfrak{a}$ -modules then  $V \otimes W$  and  $\text{Hom}_{\mathbf{C}}(V, W)$  are naturally  $\mathbf{U}(\mathfrak{a})$ -modules and  $W \otimes V \cong V \otimes W$  as  $\mathbf{U}(\mathfrak{a})$ -modules. One can also define the trivial  $\mathfrak{a}$ -module structure on  $\mathbf{C}$  and we have

$$V^{\mathfrak{a}} = \{v \in V : av = 0\} \cong \text{Hom}_{\mathfrak{a}}(\mathbf{C}, V).$$

Suppose that  $A$  is an associative commutative algebra over  $\mathbf{C}$  with unity. Then  $\mathfrak{a} \otimes A$  is canonically a Lie algebra, with the Lie bracket given by

$$[x \otimes a, y \otimes b] = [x, y] \otimes ab, \quad x, y \in \mathfrak{a}, \quad a, b \in A.$$

We shall identify  $\mathfrak{a}$  with the Lie subalgebra  $\mathfrak{a} \otimes 1$  of  $\mathfrak{a} \otimes A$ . Note that for any algebra homomorphism  $\varphi : A \rightarrow A'$  the canonical map  $1 \otimes \varphi : \mathfrak{g} \otimes A \rightarrow \mathfrak{g} \otimes A'$  is a homomorphism of Lie algebras and hence induces an algebra homomorphism  $\mathbf{U}(\mathfrak{g} \otimes A) \rightarrow \mathbf{U}(\mathfrak{g} \otimes A')$ .

**1.3.** Let  $\mathfrak{g}$  be a finite-dimensional complex simple Lie algebra with a fixed Cartan subalgebra  $\mathfrak{h}$ . Let  $\Phi$  be the corresponding root system and fix a set  $\{\alpha_i : i \in I\} \subset \mathfrak{h}^*$  (where  $I = \{1, \dots, \dim \mathfrak{h}\}$ ) of simple roots for  $\Phi$ . The root lattice  $Q$  is the  $\mathbf{Z}$ -span of the simple roots while  $Q^+$  is the  $\mathbf{Z}_+$ -span of the simple roots, and  $\Phi^+ = \Phi \cap Q^+$  denotes the set of positive roots in  $\Phi$ . Let  $\text{ht} : Q^+ \rightarrow \mathbf{Z}_+$  be the homomorphism of free semi-groups defined by setting  $\text{ht}(\alpha_i) = 1$ ,  $i \in I$ .

The restriction of the Killing form  $\kappa : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbf{C}$  to  $\mathfrak{h} \times \mathfrak{h}$  induces a non-degenerate bilinear form  $(\cdot, \cdot)$  on  $\mathfrak{h}^*$ , and we let  $\{\omega_i : i \in I\} \subset \mathfrak{h}^*$  be the fundamental weights defined by  $2(\omega_j, \alpha_i) = \delta_{i,j}(\alpha_i, \alpha_i)$ ,  $i, j \in I$ . Let  $P$  (respectively  $P^+$ ) be the  $\mathbf{Z}$  (respectively  $\mathbf{Z}_+$ ) span of the  $\{\omega_i : i \in I\}$  and note that  $Q \subseteq P$ . Given  $\lambda, \mu \in P$  we say that  $\mu \leq \lambda$  if and only if  $\lambda - \mu \in Q^+$ . Clearly  $\leq$  is a partial order on  $P$ . The set  $\Phi^+$  has a unique maximal element with respect to this order which is denoted by  $\theta$  and is called the highest root of  $\Phi^+$ . From now on, we normalize the bilinear form on  $\mathfrak{h}^*$  so that  $(\theta, \theta) = 2$ .

**1.4.** Given  $\alpha \in \Phi$ , let  $\mathfrak{g}_\alpha$  be the corresponding root space and define subalgebras  $\mathfrak{n}^\pm$  of  $\mathfrak{g}$  by

$$\mathfrak{n}^\pm = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_{\pm\alpha}.$$

We have isomorphisms of vector spaces

$$\mathfrak{g} \cong \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+, \quad \mathbf{U}(\mathfrak{g}) \cong \mathbf{U}(\mathfrak{n}^-) \otimes \mathbf{U}(\mathfrak{h}) \otimes \mathbf{U}(\mathfrak{n}^+). \quad (1.1)$$

For  $\alpha \in \Phi^+$ , fix elements  $x_\alpha^\pm \in \mathfrak{g}_{\pm\alpha}$  and  $h_\alpha \in \mathfrak{h}$  spanning a Lie subalgebra of  $\mathfrak{g}$  isomorphic to  $\mathfrak{sl}_2$ , i.e., we have

$$[h_\alpha, x_\alpha^\pm] = \pm 2x_\alpha^\pm, \quad [x_\alpha^+, x_\alpha^-] = h_\alpha,$$

and more generally, assume that the set  $\{x_\alpha^\pm : \alpha \in \Phi^+\} \cup \{h_i := h_{\alpha_i} : i \in I\}$  is a Chevalley basis for  $\mathfrak{g}$ .

**1.5.** Given an  $\mathfrak{h}$ -module  $V$ , we say that  $V$  is a weight module if

$$V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu, \quad V_\mu = \{v \in V : hv = \mu(h)v, h \in \mathfrak{h}\},$$

and we set  $\text{wt } V = \{\mu \in \mathfrak{h}^* : V_\mu \neq 0\}$ . If  $\dim V_\mu < \infty$  for all  $\mu \in \mathfrak{h}^*$ , let  $\text{ch } V$  be the character of  $V$ , namely the element of the group ring  $\mathbf{Z}[\mathfrak{h}^*]$  given by,

$$\text{ch } V = \sum_{\mu \in \mathfrak{h}^*} \dim V_\mu e(\mu),$$

where  $e(\mu) \in \mathbf{Z}[\mathfrak{h}^*]$  is the element corresponding to  $\mu \in \mathfrak{h}^*$ . Observe that for two such modules  $V_1$  and  $V_2$ , we have

$$\text{ch}(V_1 \oplus V_2) = \text{ch } V_1 + \text{ch } V_2, \quad \text{ch}(V_1 \otimes V_2) = \text{ch } V_1 \text{ch } V_2.$$

**1.6.** For  $\lambda \in P^+$ , let  $V(\lambda)$  be the left  $\mathfrak{g}$ -module generated by an element  $v_\lambda$  with defining relations:

$$hv_\lambda = \lambda(h)v_\lambda, \quad x_{\alpha_i}^+ v_\lambda = 0, \quad (x_{\alpha_i}^-)^{\lambda(h_{\alpha_i})+1} v_\lambda = 0,$$

where  $h \in \mathfrak{h}$  and  $i \in I$ . It is well-known that  $V(\lambda)$  is an irreducible weight module, and

$$\dim V(\lambda) < \infty, \quad \dim V(\lambda)_\lambda = 1, \quad \text{wt } V(\lambda) \subset \lambda - Q^+.$$

If  $V$  is a finite-dimensional irreducible  $\mathfrak{g}$ -module then there exists a unique  $\lambda \in P^+$  such that  $V$  is isomorphic to  $V(\lambda)$ . We shall say that  $V$  is a locally finite-dimensional  $\mathfrak{g}$ -module if

$$\dim \mathbf{U}(\mathfrak{g})v < \infty, \quad v \in V.$$

It is well-known that a locally finite-dimensional  $\mathfrak{g}$ -module is isomorphic to a direct sum of irreducible finite-dimensional modules, moreover

$$V_\mu \cap V^{\mathfrak{n}^+} \cong \text{Hom}_{\mathfrak{g}}(V(\mu), V), \quad \mu \in P^+.$$

**1.7.** Assume from now on that  $A$  is an associative commutative algebra over  $\mathbf{C}$  with unity. We recall the definition of the global Weyl modules. These were first introduced and studied in the case when  $A = \mathbf{C}[t, t^{-1}]$  in [6] and then later in [7] in the general case. We shall, however, follow the approach developed in [4].

**Definition.** For  $\lambda \in P^+$ , the *global Weyl module*  $W_A(\lambda)$  is the left  $\mathbf{U}(\mathfrak{g} \otimes A)$ -module generated by an element  $w_\lambda$  with defining relations,

$$(\mathfrak{n}^+ \otimes A)w_\lambda = 0, \quad hw_\lambda = \lambda(h)w_\lambda, \quad (x_{\alpha_i}^-)^{\lambda(h_{\alpha_i})+1} w_\lambda = 0,$$

where  $h \in \mathfrak{h}$  and  $i \in I$ . □

It is clear that  $W_A(\lambda)$  is not isomorphic to  $W_A(\mu)$  if  $\lambda \neq \mu$ .

Suppose that  $\varphi$  is an algebra automorphism of  $A$ . The defining ideal of  $W_A(\lambda)$  is clearly preserved by the automorphism of  $\mathbf{U}(\mathfrak{g} \otimes A)$  induced by the Lie algebra automorphism  $1 \otimes \varphi$  of  $\mathfrak{g} \otimes A$  (cf. 1.2), and we have an isomorphism of  $\mathfrak{g} \otimes A$ -modules,

$$W_A(\lambda) \cong (1 \otimes \varphi)^* W_A(\lambda). \quad (1.2)$$

**1.8.** The following construction shows immediately that  $W_A(\lambda)$  is non-zero. Given any ideal  $\mathfrak{J}$  of  $A$ , define an action of  $\mathfrak{g} \otimes A$  on  $V(\lambda) \otimes A/\mathfrak{J}$  by

$$(x \otimes a)(v \otimes b) = xv \otimes \bar{a}b, \quad x \in \mathfrak{g}, a \in A, b \in A/\mathfrak{J},$$

where  $\bar{a}$  is the canonical image of  $a$  in  $A/\mathfrak{J}$ . In particular, if  $a \notin \mathfrak{J}$  and  $h \in \mathfrak{h}$  is such that  $\lambda(h) \neq 0$  we have

$$(h \otimes a)(v_\lambda \otimes 1) = \lambda(h)v_\lambda \otimes \bar{a} \neq 0. \quad (1.3)$$

Clearly if  $\mathfrak{J} \in \text{Max } A$ , then

$$V(\lambda) \otimes A/\mathfrak{J} \cong V(\lambda)$$

as  $\mathfrak{g}$ -modules and hence  $v_\lambda \otimes 1$  generates  $V(\lambda) \otimes A/\mathfrak{J}$  as a  $\mathfrak{g}$ -module (and so also as a  $\mathfrak{g} \otimes A$ -module). Since  $v_\lambda \otimes 1$  satisfies the defining relations of  $W_A(\lambda)$ , we see that  $V(\lambda) \otimes A/\mathfrak{J}$  is a non-zero quotient of  $W_A(\lambda)$ .

**1.9.** Given a weight module  $V$  of  $\mathfrak{g} \otimes A$ , and a Lie subalgebra  $\mathfrak{a}$  of  $\mathfrak{g} \otimes A$ , set

$$V_\mu^\mathfrak{a} = V_\mu \cap V^\mathfrak{a}, \quad \mu \in \mathfrak{h}^*.$$

The following lemma is elementary.

**Lemma.** For  $\lambda \in P^+$  the module  $W_A(\lambda)$  is a locally finite-dimensional  $\mathfrak{g}$ -module and we have

$$W_A(\lambda) = \bigoplus_{\eta \in Q^+} W_A(\lambda)_{\lambda-\eta}$$

which in particular means that  $\text{wt } W_A(\lambda) \subset \lambda - Q^+$ . If  $V$  is a  $\mathfrak{g} \otimes A$ -module which is locally finite-dimensional as a  $\mathfrak{g}$ -module then we have an isomorphism of vector spaces,

$$\text{Hom}_{\mathfrak{g} \otimes A}(W_A(\lambda), V) \cong V_\lambda^{\mathfrak{n}^+ \otimes A}. \quad \square$$

**1.10.** The weight spaces  $W_A(\lambda)_{\lambda-\eta}$  are not necessarily finite-dimensional, and to understand them, we proceed as follows. It is easily checked that one can regard  $W_A(\lambda)$  as a right module for  $\mathbf{U}(\mathfrak{h} \otimes A)$  by setting

$$(uw_\lambda)(h \otimes a) = u(h \otimes a)w_\lambda, \quad u \in \mathbf{U}(\mathfrak{g} \otimes A), \quad h \in \mathfrak{h}, \quad a \in A.$$

Since  $\mathbf{U}(\mathfrak{h} \otimes A)$  is commutative, the algebra  $\mathbf{A}_\lambda$  defined by

$$\mathbf{A}_\lambda = \mathbf{U}(\mathfrak{h} \otimes A) / \text{Ann}_{\mathbf{U}(\mathfrak{h} \otimes A)} w_\lambda$$

is a commutative associative algebra. It follows that  $W_A(\lambda)$  is a  $(\mathfrak{g} \otimes A, \mathbf{A}_\lambda)$ -bimodule and that for all  $\eta \in Q^+$ , the weight space  $W_A(\lambda)_{\lambda-\eta}$  is a right  $\mathbf{A}_\lambda$ -module. Clearly

$$W_A(\lambda)_\lambda \cong_{\mathbf{A}_\lambda} \mathbf{A}_\lambda,$$

where we regard  $\mathbf{A}_\lambda$  as a right  $\mathbf{A}_\lambda$ -module through the right multiplication. The following is immediate.

**Lemma.** For  $\lambda \in P^+$ ,  $\eta \in Q^+$  the subspaces  $W_A(\lambda)_{\lambda-\eta}^{\mathfrak{n}^+}$  and  $W_A(\lambda)_{\lambda-\eta}^{\mathfrak{n}^+ \otimes A}$  are  $\mathbf{A}_\lambda$ -submodules of  $W_A(\lambda)$  and we have

$$W_A(\lambda)_\lambda^{\mathfrak{n}^+} = W_A(\lambda)_\lambda^{\mathfrak{n}^+ \otimes A} = W_A(\lambda)_\lambda.$$

Further, if  $\mu \in P^+$ , the space  $\text{Hom}_{\mathfrak{g} \otimes A}(W_A(\mu), W_A(\lambda))$  has the natural structure of a right  $\mathbf{A}_\lambda$ -module and

$$\text{Hom}_{\mathfrak{g} \otimes A}(W_A(\mu), W_A(\lambda)) \cong_{\mathbf{A}_\lambda} W_A(\lambda)_\mu^{\mathfrak{n}^+ \otimes A}. \quad \square$$

**1.11.**

**Lemma.** For  $\lambda \in P^+$ ,  $\mathfrak{a} \in \mathbf{A}_\lambda$  the assignment  $w_\lambda \rightarrow w_\lambda \mathfrak{a}$  extends to a well-defined homomorphism  $W_A(\lambda) \rightarrow W_A(\lambda)$  of  $\mathfrak{g} \otimes A$ -modules and we have

$$\text{Hom}_{\mathfrak{g} \otimes A}(W_A(\lambda), W_A(\lambda)) \cong_{\mathbf{A}_\lambda} \mathbf{A}_\lambda \cong_{\mathbf{A}_\lambda} W_A(\lambda)_\lambda^{\mathfrak{n}^+ \otimes A}.$$

*Proof.* Since  $W_A(\lambda)$  is a  $(\mathfrak{g} \otimes A, \mathbf{A}_\lambda)$ -bimodule, it follows that  $w_\lambda \mathfrak{a}$  satisfies the defining relations of  $W_A(\lambda)$  which yields the first statement of the Lemma. For the second, let  $\pi : W_A(\lambda) \rightarrow W_A(\lambda)$  be a nonzero  $\mathfrak{g} \otimes A$ -module map. Since  $W_A(\lambda)_\lambda = \mathbf{U}(\mathfrak{h} \otimes A)w_\lambda$ , there exists  $u_\pi \in \mathbf{U}(\mathfrak{h} \otimes A)$  such that  $\pi(w_\lambda) = u_\pi w_\lambda$ . Since  $\pi$  is non-zero, the image  $\tilde{u}_\pi$  of  $u_\pi$  in  $\mathbf{A}_\lambda$  is non-zero. Thus, we obtain a well-defined map  $\text{Hom}_{\mathfrak{g} \otimes A}(W_A(\lambda), W_A(\lambda)) \rightarrow \mathbf{A}_\lambda$  given by  $\pi \mapsto \tilde{u}_\pi$ . It is clear that the map is an isomorphism of right  $\mathbf{A}_\lambda$ -modules and the lemma is proved.  $\square$

## 2. THE MAIN RESULTS

**2.1.** For  $\mathbf{s} = (s_i)_{i \in I} \in \mathbf{Z}_+^I$ , set

$$\mathbf{A}_{\mathbf{s}} = \bigotimes_{i \in I} \mathbf{A}_{\omega_i}^{\otimes s_i}, \quad W_A(\mathbf{s}) = \bigotimes_{i \in I} W_A(\omega_i)^{\otimes s_i}, \quad w_{\mathbf{s}} = \bigotimes_{i \in I} w_{\omega_i}^{\otimes s_i}, \quad (2.1)$$

where all tensor products are taken in the same (fixed) order. Given  $k, \ell \in \mathbf{Z}_+$  let  $\mathcal{R}_{k, \ell}$  be the algebra of polynomials  $\mathbf{C}[t_1^{\pm 1}, \dots, t_k^{\pm 1}, u_1, \dots, u_\ell]$  with the convention that if  $k = 0$  (respectively  $\ell = 0$ ),  $\mathcal{R}_{0, \ell}$  (respectively  $\mathcal{R}_{k, 0}$ ) is just the ring of polynomials (respectively Laurent polynomials) in  $\ell$  (respectively  $k$ ) variables.

**2.2.** The main result of this paper is the following.

**Theorem 2.** Assume that  $A = \mathcal{R}_{k, \ell}$  for some  $k, \ell \in \mathbf{Z}_+$ . For all  $\mathbf{s} \in \mathbf{Z}_+^n$  and  $\mu \in P^+$ , we have

$$\mathrm{Hom}_{\mathfrak{g} \otimes A}(W_A(\mu), W_A(\mathbf{s})) \cong_{\mathbf{A}_{\mathbf{s}}} W_A(\mathbf{s})_{\mu}^{n^+ \otimes A} = \left( \bigotimes_{i \in I} (W_A(\omega_i)^{n^+ \otimes A})^{\otimes s_i} \right)_{\mu}. \quad (2.2)$$

In the case when  $\mathfrak{g}$  is a classical simple Lie algebra, we can make (2.2) more precise. Let  $I_0$  be the set of  $i \in I$  such that  $\alpha_i$  occurs in  $\theta$  with the coefficient  $2/(\alpha_i, \alpha_i)$ . In particular,  $I_0 = I$  for  $\mathfrak{g}$  of type  $A$  or  $C$ . Given  $\mathbf{s} = (s_i)_{i \in I} \in \mathbf{Z}_+^I$  and  $\lambda \in P^+$ , let  $\mathbf{c}_{\mathbf{s}}(\lambda) \in \mathbf{Z}_+$  be the coefficient of  $e(\lambda)$  in

$$\prod_{i \in I_0} e(\omega_i)^{s_i} \prod_{i \notin I_0} \left( \sum_{0 \leq j \leq i/2} \binom{j+k-1}{j} e(\omega_{i-2j}) \right)^{s_i},$$

where  $\omega_0 = 0$ .

**Corollary.** Let  $\lambda \in P^+$  and  $\mathbf{s} \in \mathbf{Z}_+^I$ . Assume either that  $\mathfrak{g}$  is not an exceptional Lie algebra or that  $s_i = 0$  if  $i \notin I_0$ . We have

$$\mathrm{Hom}_{\mathfrak{g} \otimes A}(W_A(\lambda), W_A(\mathbf{s})) \cong_{\mathbf{A}_{\mathbf{s}}} \mathbf{A}_{\mathbf{s}}^{\oplus \mathbf{c}_{\mathbf{s}}(\lambda)},$$

where we use the convention that  $\mathbf{A}_{\mathbf{s}}^{\oplus \mathbf{c}_{\mathbf{s}}(\lambda)} = 0$  if  $\mathbf{c}_{\mathbf{s}}(\lambda) = 0$ .

**2.3.** Our next result is the following. Recall from Lemma 1.11 that for all  $\lambda \in P^+$  we have  $\mathrm{Hom}_{\mathfrak{g} \otimes A}(W_A(\lambda), W_A(\lambda)) \cong_{\mathbf{A}_{\lambda}} \mathbf{A}_{\lambda}$ .

**Theorem 3.** Let  $A$  be the ring  $\mathcal{R}_{0,1}$  or  $\mathcal{R}_{1,0}$ . For all  $\mu = \sum_{i \in I} s_i \omega_i \in P^+$  with  $s_i = 0$  if  $i \notin I_0$ , and all  $\lambda \in P^+$ , we have

$$\mathrm{Hom}_{\mathfrak{g} \otimes A}(W_A(\lambda), W_A(\mu)) = 0, \quad \text{if } \lambda \neq \mu.$$

Further, any non-zero element of  $\mathrm{Hom}_{\mathfrak{g} \otimes A}(W_A(\mu), W_A(\mu))$  is injective. An analogous result holds when  $A = \mathcal{R}_{k, \ell}$ ,  $k, \ell \in \mathbf{Z}_+$ ,  $\mathfrak{g} \cong \mathfrak{sl}_{n+1}$  and  $\mu = s\omega_1$ .



**2.4.** We now make some comments on the various restrictions in the main results. The proof of Theorem 2 relies on an explicit construction of the fundamental global Weyl modules in terms of certain finite-dimensional modules called the fundamental local Weyl modules. A crucial ingredient of this construction is a natural bialgebra structure of  $\mathcal{R}_{k,\ell}$ . The proof of Corollary 2.2 depends on a deeper understanding of the  $\mathfrak{g}$ -module structure of the local fundamental Weyl modules. These results are unavailable for the exceptional Lie algebras when  $k + \ell > 1$ . In the case when  $k + \ell = 1$ , the structure of these modules for the exceptional algebras is known as a consequence of the work of many authors on the Kirillov–Reshetikhin conjecture (see [3] for extensive references on the subject). Hence, a precise statement of Corollary 2.2 could be made when  $k + \ell = 1$  in a case by case and in a not very compact fashion. The interested reader is referred to [9] and [11].

**2.5.** Before discussing Theorem 3, we make the following conjecture.

**Conjecture.** Let  $A = \mathcal{R}_{k,\ell}$  for some  $k, \ell \in \mathbf{Z}_+$ . Then for all  $\lambda \in P^+$  and  $\mathbf{s} \in \mathbf{Z}_+^n$ , any non-zero element of  $\text{Hom}_{\mathfrak{g} \otimes A}(W_A(\lambda), W_A(\mathbf{s}))$  is injective.

The proof of Theorem 3 will rely on the fact that this conjecture is true (see Section 5 of this paper) when  $k + \ell = 1$  and  $s_i = 0$  if  $i \notin I_0$  as well as on the fact that the fundamental local Weyl module is irreducible as a  $\mathfrak{g}$ -module if  $i \in I_0$ . We shall prove in Section 5 using Corollary 2.2 and the work of [7] that the conjecture is also true when  $\mathfrak{g} = \mathfrak{sl}_{r+1}$  and  $\lambda = s\omega_1$ . Remark 6.1 of this paper shows that  $\text{Hom}_{\mathfrak{g} \otimes A}(W_A(\lambda), W_A(\mu))$  can be non-zero if we remove the restriction on  $\mu$ .

**2.6.** Finally, we make some remarks on quantum analogs of this result. In the case of the quantum loop algebra, one also has analogous notions of global and local Weyl modules which were defined in [6], and one can construct the global fundamental Weyl module from the local Weyl module in a way analogous to the one given in this paper for  $A = \mathbf{C}[t, t^{-1}]$ . It was shown in [6] for the quantum loop algebra of  $\mathfrak{sl}_2$  that the canonical map from the global Weyl module into the tensor product of fundamental global Weyl modules is injective. For the general quantum loop algebra, this was established by Beck and Nakajima ([1]) using crystal and global bases. They also describe the space of extremal weight vectors in the tensor product of quantum fundamental global Weyl modules.

### 3. THE ALGEBRA $\mathbf{A}_\lambda$ AND THE LOCAL WEYL MODULES

In this section we recall some necessary results from [4] and also the definition and elementary properties of local Weyl modules.

**3.1.** For  $r \in \mathbf{Z}_+$  the symmetric group  $S_r$  acts naturally on  $A^{\otimes r}$  and on  $(\text{Max } A)^{\times r}$  and we let  $(A^{\otimes r})^{S_r}$  be the corresponding ring of invariants and  $(\text{Max } A)^{\times r}/S_r$  the set of orbits. If  $r = r_1 + \cdots + r_n$ , then we regard  $S_{r_1} \times \cdots \times S_{r_n}$  as a subgroup of  $S_r$  in the canonical way, i.e.  $S_{r_1}$  permutes the first  $r_1$  letters,  $S_{r_2}$  the next  $r_2$  letters and so on. Given  $\lambda = \sum_{i \in I} r_i \omega_i \in P^+$ ,



set

$$r_\lambda = \sum_{i \in I} r_i, \quad S_\lambda = S_{r_1} \times \cdots \times S_{r_n}, \quad \mathbb{A}_\lambda = (A^{\otimes r_\lambda})^{S_\lambda}, \quad (3.1)$$

$$\text{Max } \mathbb{A}_\lambda = (\text{Max } A)^{r_\lambda} / S_\lambda. \quad (3.2)$$

The algebra  $\mathbb{A}_\lambda$  is generated by elements of the form

$$\text{sym}_\lambda^i(a) = 1^{\otimes(r_1+\cdots+r_{i-1})} \otimes \left( \sum_{k=0}^{r_i-1} 1^{\otimes k} \otimes a \otimes 1^{\otimes(r_i-k-1)} \right) \otimes 1^{\otimes(r_{i+1}+\cdots+r_n)}, \quad a \in A, i \in I. \quad (3.3)$$

The following was proved in [4, Theorem 4].

**Proposition.** *Let  $A$  be a finitely generated commutative associative algebra over  $\mathbf{C}$  with trivial Jacobson radical. Then the homomorphism of associative algebras  $\mathbf{U}(\mathfrak{h} \otimes A) \rightarrow \mathbb{A}_\lambda$  defined by*

$$h_i \otimes a \mapsto \text{sym}_\lambda^i(a), \quad i \in I, \quad a \in A$$

*induces an isomorphism of algebras  $\text{sym}_\lambda : \mathbf{A}_\lambda \xrightarrow{\sim} \mathbb{A}_\lambda$ . In particular, if  $A$  is a finitely generated integral domain then  $\mathbf{A}_\lambda$  is isomorphic to an integral subdomain of  $\mathbf{A}_\mathbf{r}$ .*  $\square$

**3.2.** For  $\lambda, \mu \in P^+$ , it is clear that the tensor product  $W_A(\lambda) \otimes W_A(\mu)$  has the natural structure of a  $(\mathfrak{g} \otimes A, \mathbf{A}_\lambda \otimes \mathbf{A}_\mu)$ -module. We recall from [4] that, in fact, there exists a  $(\mathfrak{g} \otimes A, \mathbf{A}_{\lambda+\mu})$ -bimodule structure on  $W_A(\lambda) \otimes W_A(\mu)$ .

It is clear from Definition 1.7 that the assignment  $w_{\lambda+\mu} \mapsto w_\lambda \otimes w_\mu$  defines a homomorphism  $\tau_{\lambda,\mu} : W_A(\lambda + \mu) \rightarrow W_A(\lambda) \otimes W_A(\mu)$  of  $\mathfrak{g} \otimes A$ -modules. The restriction of this map to  $W_A(\lambda + \mu)_{\lambda+\mu}$  induces a homomorphism of algebras  $\mathbf{A}_{\lambda+\mu} \rightarrow \mathbf{A}_\lambda \otimes \mathbf{A}_\mu$  as follows. Consider the restriction of the comultiplication  $\Delta$  of  $\mathbf{U}(\mathfrak{g} \otimes A)$  to  $\mathbf{U}(\mathfrak{h} \otimes A)$ . It is not hard to see that

$$\text{Ann}_{\mathbf{U}(\mathfrak{h} \otimes A) \otimes \mathbf{U}(\mathfrak{h} \otimes A)}(w_\lambda \otimes w_\mu) \subset \text{Ann}_{\mathbf{U}(\mathfrak{h} \otimes A)} w_\lambda \otimes \mathbf{U}(\mathfrak{h} \otimes A) + \mathbf{U}(\mathfrak{h} \otimes A) \otimes \text{Ann}_{\mathbf{U}(\mathfrak{h} \otimes A)} w_\mu,$$

and hence we have

$$\Delta(\text{Ann}_{\mathbf{U}(\mathfrak{h} \otimes A)}(w_{\lambda+\mu})) \subset \text{Ann}_{\mathbf{U}(\mathfrak{h} \otimes A)} w_\lambda \otimes \mathbf{U}(\mathfrak{h} \otimes A) + \mathbf{U}(\mathfrak{h} \otimes A) \otimes \text{Ann}_{\mathbf{U}(\mathfrak{h} \otimes A)} w_\mu.$$

It is now immediate that the comultiplication  $\Delta : \mathbf{U}(\mathfrak{h} \otimes A) \rightarrow \mathbf{U}(\mathfrak{h} \otimes A) \otimes \mathbf{U}(\mathfrak{h} \otimes A)$  induces a homomorphism of algebras  $\Delta_{\lambda,\mu} : \mathbf{A}_{\lambda+\mu} \rightarrow \mathbf{A}_\lambda \otimes \mathbf{A}_\mu$ . This endows any right  $\mathbf{A}_\lambda \otimes \mathbf{A}_\mu$ -module (hence, in particular,  $W_A(\lambda) \otimes W_A(\mu)$ ) with the structure of a right  $\mathbf{A}_{\lambda+\mu}$ -module. It was shown in [4] that  $\tau_{\lambda,\mu}$  is then a homomorphism of  $(\mathfrak{g} \otimes A, \mathbf{A}_{\lambda+\mu})$ -bimodules. Summarizing, we have

**Lemma.** *Let  $\lambda_s \in P^+$ ,  $1 \leq s \leq k$  and let  $\lambda = \sum_{s=1}^k \lambda_s$ . The natural map  $W_A(\lambda) \rightarrow W_A(\lambda_1) \otimes \cdots \otimes W_A(\lambda_k)$  given by  $w_\lambda \mapsto w_{\lambda_1} \otimes \cdots \otimes w_{\lambda_k}$  is a homomorphism of  $(\mathfrak{g} \otimes A, \mathbf{A}_\lambda)$ -bimodules.*  $\square$

**3.3.** For  $\lambda \in P^+$ , let  $\text{mod } \mathbf{A}_\lambda$  be the category of left  $\mathbf{A}_\lambda$ -modules and let  $\mathbf{W}_A^\lambda$  be the right exact functor from  $\text{mod } \mathbf{A}_\lambda$  to the category of  $\mathfrak{g} \otimes A$ -modules given on objects by

$$\mathbf{W}_A^\lambda M = W_A(\lambda) \otimes_{\mathbf{A}_\lambda} M, \quad M \in \text{mod } \mathbf{A}_\lambda.$$

Clearly  $\mathbf{W}_A^\lambda M$  is a weight module for  $\mathfrak{g}$  and we have isomorphisms of vector spaces

$$\begin{aligned} (\mathbf{W}_A^\lambda M)_{\lambda-\eta} &\cong (W_A(\lambda))_{\lambda-\eta} \otimes_{\mathbf{A}_\lambda} M, & \eta \in Q^+, \\ (\mathbf{W}_A^\lambda M)_\lambda &\cong (W_A(\lambda)_\lambda) \otimes_{\mathbf{A}_\lambda} M \cong w_\lambda \otimes_{\mathbf{C}} M. \end{aligned}$$

Moreover,  $\mathbf{W}_A^\lambda M$  is generated as a  $\mathfrak{g} \otimes A$ -module by the space  $w_\lambda \otimes_{\mathbf{C}} M$  and

$$\mathbf{W}_A^\lambda M \cong \mathbf{W}_A^\mu N \iff \lambda = \mu, \quad M \cong_{\mathbf{A}_\lambda} N.$$

The isomorphism classes of simple objects of  $\text{mod } \mathbf{A}_\lambda$  are given by the maximal ideals of  $\mathbf{A}_\lambda$ . Given  $\mathbf{I} \in \text{Max } \mathbf{A}_\lambda$ , the quotient  $\mathbf{A}_\lambda/\mathbf{I}$  is a simple object of  $\text{mod } \mathbf{A}_\lambda$  and has dimension one. The  $\mathfrak{g} \otimes A$ -modules  $\mathbf{W}_A^\lambda \mathbf{A}_\lambda/\mathbf{I}$  are called the local Weyl modules and when  $\lambda = \omega_i$ ,  $i \in I$  we call them the fundamental local Weyl modules. It follows that  $(\mathbf{W}_A^\lambda \mathbf{A}_\lambda/\mathbf{I})_\lambda$  is also a one-dimensional vector space spanned by

$$w_{\lambda, \mathbf{A}_\lambda/\mathbf{I}} = w_\lambda \otimes 1.$$

We note the following corollary.

**Corollary.** *Suppose that  $M \in \text{mod } \mathbf{A}_\lambda$  is finite-dimensional. Then  $\mathbf{W}_A^\lambda M$  is finite-dimensional. In particular, the local Weyl modules are finite-dimensional and have a unique irreducible quotient  $\mathbf{V}_A^\lambda M$ .*

**3.4.** We note the following consequence of Proposition 3.1.

**Lemma.** *For  $i \in I$ , we have  $\mathbf{A}_{\omega_i} \cong A$  and  $W_A(\omega_i)$  is a finitely generated right  $A$ -module. The fundamental local Weyl modules are given by*

$$\mathbf{W}_A^{\omega_i} A/\mathfrak{J} = W_A(\omega_i) \otimes_A A/\mathfrak{J}, \quad \mathfrak{J} \in \text{Max } A.$$

*In particular, we have*

$$(h \otimes a)w_{\omega_i, A/\mathfrak{J}} = 0, \quad (h \otimes b)w_{\omega_i, A/\mathfrak{J}} = \omega_i(h)w_{\omega_i} \otimes \bar{b}, \quad h \in \mathfrak{h}, a \in \mathfrak{J}, b \in A. \quad \square$$

**3.5.** The following lemma is a special case of a result proved in [4] and we include the proof in this case for the reader's convenience.

**Lemma.** *Let  $A$  be a finitely generated, commutative associative algebra. For  $\mathfrak{J} \in \text{Max } A$  there exists  $N \in \mathbf{Z}_+$  such that for all  $i \in I$ ,*

$$(\mathfrak{g} \otimes \mathfrak{J}^N) \mathbf{W}_A^{\omega_i} A/\mathfrak{J} = 0. \quad (3.4)$$

*Proof.* Since  $\mathfrak{g}$  is a simple Lie algebra (3.4) follows if we show that there exists  $N \geq 0$  such that

$$(x_\theta^- \otimes \mathfrak{J}^N) \mathbf{W}_A^{\omega_i} A/\mathfrak{J} = 0.$$

Since

$$\mathbf{W}_A^{\omega_i} A/\mathfrak{J} = \mathbf{U}(\mathfrak{n}^- \otimes A)w_{\omega_i, A/\mathfrak{J}}, \quad [x_\theta^-, \mathfrak{n}^-] = 0,$$

it is enough to show that there exists  $N \geq 0$  such that

$$(x_\theta^- \otimes \mathfrak{J}^N)w_{\omega_i, A/\mathfrak{J}} = 0.$$

We claim that is a consequence of showing that for  $j \in I$  there exists  $N_j \in \mathbf{Z}_+$ , with

$$(x_{\alpha_j}^- \otimes \mathfrak{J}^{N_j})w_{\omega_i, A/\mathfrak{J}} = 0. \quad (3.5)$$

To prove the claim, write  $x_\theta^- = [x_{\alpha_{i_1}}^- [\cdots [x_{\alpha_{i_{p-1}}}^-, x_{\alpha_{i_p}}^-] \cdots]]$  for some  $i_1, \dots, i_p \in I$  and take  $N = \sum_{j=1}^p N_{i_j}$ . To prove (3.5), observe first that for  $j \neq i \in I$ ,  $k \in I$  and for all  $a \in A$

$$x_{\alpha_k}^+(x_{\alpha_j}^- \otimes a)w_{\omega_i, A/\mathfrak{J}} = \delta_{k,j}(h_j \otimes a)w_{\omega_i, \mathfrak{J}} = 0,$$

by Lemma 3.4 and the defining relations of  $W_A(\omega_i)$ . Thus,  $(x_{\alpha_j}^- \otimes a)w_{\omega_i, A/\mathfrak{J}} \in (\mathbf{W}_A^{\omega_i} A/\mathfrak{J})^{\mathfrak{n}^+}$  and since  $\omega_i - \alpha_j \notin P^+$  we conclude that

$$(x_{\alpha_j}^- \otimes A)w_{\omega_i, A/\mathfrak{J}} = 0, \quad j \neq i.$$

If  $j = i$ , then

$$0 = (x_{\alpha_i}^+ \otimes a)(x_{\alpha_i}^-)^2 w_{\omega_i} = 2((x_{\alpha_i}^- \otimes 1)(h_i \otimes a) - (x_{\alpha_i}^- \otimes a))w_{\omega_i}.$$

By Lemma 3.4, we have  $(h_i \otimes a)w_{\omega_i, A/\mathfrak{J}} = 0$  if  $a \in \mathfrak{J}$ , and so we get

$$(x_{\alpha_i}^- \otimes a)w_{\omega_i, A/\mathfrak{J}} = 0, \quad a \in \mathfrak{J}$$

and (3.5) is established.  $\square$

#### 4. FUNDAMENTAL GLOBAL WEYL MODULES

In this section we establish the main tool for proving Theorem 2. It is not, in general, clear how (or even if it is possible) to reconstruct the global Weyl module from a local Weyl module. The main result of this section is that it is possible to do so when  $\lambda = \omega_i$  and  $A = \mathcal{R}_{k, \ell}$  for some  $k, \ell \in \mathbf{Z}_+$ .

**4.1.** We begin with a general construction. The Lie algebra  $(\mathfrak{g} \otimes A) \otimes A$  acts naturally on  $V \otimes A$  for any  $\mathfrak{g} \otimes A$ -module  $V$ . Suppose that  $A$  is a bialgebra with the comultiplication  $\mathbf{h} : A \rightarrow A \otimes A$ . (It is useful to recall that  $A$  is a commutative associative algebra with identity). Then the comultiplication map  $\mathbf{h}$  induces a homomorphism of Lie algebras  $1 \otimes \mathbf{h} : \mathfrak{g} \otimes A \rightarrow \mathfrak{g} \otimes A \otimes A$  (cf. 1.2) and thus a  $\mathfrak{g} \otimes A$ -module structure on  $V \otimes A$ . Explicitly, the  $(\mathfrak{g} \otimes A, A)$ -bimodule structure on  $V \otimes A$  is given by the following formulas:

$$(x \otimes a)(v \otimes b) = \sum_s (x \otimes a'_s)v \otimes a''_s b, \quad (v \otimes b)a = v \otimes ba, \quad v \in V, a, b \in A,$$

where  $\mathbf{h}(a) = \sum_s a'_s \otimes a''_s$ . We denote this bimodule by  $(V \otimes A)_{\mathbf{h}}$  and observe that it is a free right  $A$ -module of rank equal to  $\dim_{\mathbf{C}} V$ . It is trivial to see that  $(V \otimes A)_{\mathbf{h}}$  is a weight module for  $\mathfrak{g} \otimes A$  if  $V$  is a weight module for  $\mathfrak{g} \otimes A$  and that

$$((V \otimes A)_{\mathbf{h}})_{\mu} = V_{\mu} \otimes A, \quad V^{\mathfrak{n}^+ \otimes A} \otimes A \subset (V \otimes A)_{\mathbf{h}}^{\mathfrak{n}^+ \otimes A}.$$

Moreover, if  $V_1, V_2$  are  $\mathfrak{g} \otimes A$ -modules, one has a natural inclusion

$$\mathrm{Hom}_{\mathfrak{g} \otimes A}(V_1, V_2) \hookrightarrow \mathrm{Hom}_{\mathfrak{g} \otimes A}((V_1 \otimes A)_{\mathbf{h}}, (V_2 \otimes A)_{\mathbf{h}}), \quad \eta \mapsto \eta \otimes 1. \quad (4.1)$$

In particular, if  $V$  is reducible, then  $(V \otimes A)_{\mathbf{h}}$  is also a reducible  $(\mathfrak{g} \otimes A)$ -module.

**4.2.** Let  $\mathbf{h}_{k,\ell} : \mathcal{R}_{k,\ell} \rightarrow \mathcal{R}_{k,\ell} \otimes \mathcal{R}_{k,\ell}$  be the comultiplication given by,

$$\mathbf{h}_{k,\ell}(t_s^{\pm 1}) = t_s^{\pm 1} \otimes t_s^{\pm 1}, \quad \mathbf{h}_{k,\ell}(u_r) = u_r \otimes 1 + 1 \otimes u_r,$$

where  $1 \leq s \leq k$  and  $1 \leq r \leq \ell$ . Any monomial  $\mathbf{m} \in \mathcal{R}_{k,\ell}$  can be written uniquely as a product of monomials

$$\mathbf{m} = \mathbf{m}_u \mathbf{m}_t, \quad \mathbf{m}_t \in \mathbf{C}[t_1^{\pm 1}, \dots, t_k^{\pm 1}], \mathbf{m}_u \in \mathbf{C}[u_1, \dots, u_\ell].$$

Set  $\deg t_s^{\pm 1} = \pm 1$  and  $\deg u_r = 1$  for  $1 \leq s \leq k$ ,  $1 \leq r \leq \ell$  and let  $\deg_t \mathbf{m}$  (respectively,  $\deg_u \mathbf{m}$ ) be the total degree of  $\mathbf{m}_t$  (respectively,  $\mathbf{m}_u$ ) and for any  $f \in A$  define  $\deg_t f$  and  $\deg_u f$  in the obvious way. The next lemma is elementary.

**Lemma.** Let  $\mathbf{m} = \mathbf{m}_t \mathbf{m}_u$  be a monomial in  $\mathcal{R}_{k,\ell}$ . Then  $\mathbf{m}_t \in \mathcal{R}_{k,\ell}^\times$ ,  $\mathbf{h}_{k,\ell}(\mathbf{m}_t) = \mathbf{m}_t \otimes \mathbf{m}_t$  and

$$\mathbf{h}_{k,\ell}(\mathbf{m}) = \mathbf{m} \otimes \mathbf{m}_t + \sum_q \mathbf{m}'_{u,q} \mathbf{m}_t \otimes \mathbf{m}''_{u,q} \mathbf{m}_t = \mathbf{m}_t \otimes \mathbf{m} + \sum_q \mathbf{m}''_{u,q} \mathbf{m}_t \otimes \mathbf{m}'_{u,q} \mathbf{m}_t, \quad (4.2)$$

where  $\mathbf{m}'_{u,q}, \mathbf{m}''_{u,q}$  are (scalar multiples of) monomials in the  $u_r$ ,  $1 \leq r \leq \ell$ , such that if  $\mathbf{m}'_{u,q} \neq 0$ ,  $\mathbf{m}''_{u,q} \neq 0$ , then  $\deg_u \mathbf{m}'_{u,q} < \deg_u \mathbf{m}$  and  $\deg_u \mathbf{m}'_{u,q} + \deg_u \mathbf{m}''_{u,q} = \deg_u \mathbf{m}$ .  $\square$

**4.3.** For the rest of the section  $A$  denotes the algebra  $\mathcal{R}_{k,\ell}$  for some  $k, \ell \in \mathbf{Z}_+$  and  $\mathfrak{J}$  the ideal of  $A$  generated by the elements  $\{t_1 - 1, \dots, t_k - 1, u_1, \dots, u_\ell\}$ .

Suppose that  $\mathfrak{J} \in \text{Max } \mathcal{R}_{k,\ell}$ . It is clear that there exists an algebra automorphism  $\varphi : \mathcal{R}_{k,\ell} \rightarrow \mathcal{R}_{k,\ell}$  such that  $\varphi(\mathfrak{J}) = \mathfrak{J}$ . As a consequence, we have an induced isomorphism of  $\mathfrak{g} \otimes A$ -modules,

$$\mathbf{W}_A^{\omega_i} A / \mathfrak{J} \cong (1 \otimes \varphi)^* \mathbf{W}_A^{\omega_i} A / \mathfrak{J}. \quad (4.3)$$

Moreover, if we set

$$\mathbf{h}_{k,\ell}^\varphi = (\varphi \otimes \varphi) \circ \mathbf{h}_{k,\ell} \circ \varphi^{-1} : A \rightarrow A \otimes A,$$

then  $\mathbf{h}_{k,\ell}^\varphi$  also defines a bialgebra structure on  $A$  and we have an isomorphism of  $\mathfrak{g} \otimes A$ -modules

$$(1 \otimes \varphi)^* (\mathbf{W}_A^{\omega_i} A / \mathfrak{J} \otimes A)_{\mathbf{h}_{k,\ell}} \cong (\mathbf{W}_A^{\omega_i} A / \mathfrak{J} \otimes A)_{\mathbf{h}_{k,\ell}^\varphi}. \quad (4.4)$$

This becomes an isomorphism of  $(\mathfrak{g} \otimes A, A)$ -bimodules if we twist the right  $A$ -module structure of  $(\mathbf{W}_A^{\omega_i} A / \mathfrak{J} \otimes A)_{\mathbf{h}_{k,\ell}^\varphi}$  by  $\varphi$ .

**4.4.** We now reconstruct the global fundamental Weyl module from a local one.

**Proposition.** For all  $i \in I$ , the assignment  $w_{\omega_i} \mapsto w_{\omega_i, A/\mathfrak{J}} \otimes 1$  defines an isomorphism of  $(\mathfrak{g} \otimes A, A)$ -bimodules

$$W_A(\omega_i) \xrightarrow{\cong} (\mathbf{W}_A^{\omega_i} A / \mathfrak{J} \otimes A)_{\mathbf{h}_{k,\ell}}.$$

**Remark.** It is clear from (4.4) and Section 1.7 that one can work with an arbitrary ideal  $\mathfrak{J}$  provided that  $\mathbf{h}_{k,\ell}$  is replaced by  $\mathbf{h}_{k,\ell}^\varphi$ , where  $\varphi$  is the unique automorphism of  $A$  such that  $\varphi(\mathfrak{J}) = \mathfrak{J}$ .

*Proof.* The element  $w_{\omega_i, A/\mathfrak{J}} \otimes 1 \in (\mathbf{W}_A^{\omega_i} A / \mathfrak{J} \otimes A)_{\mathbf{h}}$  satisfies the relations in Definition 1.7 and hence the assignment  $w_{\omega_i} \mapsto w_{\omega_i, A/\mathfrak{J}} \otimes 1$  defines a homomorphism of  $\mathfrak{g} \otimes A$ -modules

$$\mathbf{p} : W_A(\omega_i) \rightarrow (\mathbf{W}_A^{\omega_i} A / \mathfrak{J} \otimes A)_{\mathbf{h}_{k,\ell}}.$$

We begin by proving that  $\mathbf{p}$  is a homomorphism of right  $A$ -modules. Using Lemma 3.4 and the definition of the right module structure on  $W_A(\omega_i)$  we see that

$$\mathbf{p}((uw_{\omega_i})a) = \mathbf{p}(u(h_i \otimes a)w_{\omega_i}) = u(h_i \otimes a)(w_{\omega_i, A/\mathfrak{J}} \otimes 1), \quad u \in \mathbf{U}(\mathfrak{g} \otimes A), a \in A.$$

This shows that it is enough to prove that for any monomial  $\mathbf{m}$  in  $A$ , we have

$$(h_i \otimes \mathbf{m})(w_{\omega_i, A/\mathfrak{J}} \otimes 1) = w_{\omega_i, A/\mathfrak{J}} \otimes \mathbf{m}. \quad (4.5)$$

Write  $\mathbf{m} = \mathbf{m}_t \mathbf{m}_u$  and observe that  $\mathbf{m}_t - 1 \in \mathfrak{J}$  while

$$\deg_u \mathbf{m} > 0 \implies \mathbf{m} \in \mathfrak{J}.$$

Using (4.2) we get

$$\mathbf{h}_{k, \ell}(\mathbf{m}) - 1 \otimes \mathbf{m} \in \mathfrak{J} \otimes A$$

and since  $(h_i \otimes \mathfrak{J} \otimes A)(w_{\omega_i, A/\mathfrak{J}} \otimes 1) = 0$  by Lemma 3.4, we have established (4.5).

To prove that  $\mathbf{p}$  is surjective we must show that

$$\mathbf{U}(\mathfrak{g} \otimes A)(w_{\omega_i, A/\mathfrak{J}} \otimes 1) = (\mathbf{W}_A^{\omega_i} A/\mathfrak{J} \otimes A)_{\mathbf{h}_{k, \ell}},$$

and the remarks in Section 4.1 show that it is enough to prove

$$(\mathbf{W}_A^{\omega_i} A/\mathfrak{J})_{\omega_i - \eta} \otimes A \subset \mathbf{U}(\mathfrak{g} \otimes A)(w_{\omega_i, A/\mathfrak{J}} \otimes 1), \quad \eta \in Q^+. \quad (4.6)$$

The argument is by induction on  $\text{ht } \eta$ , the induction base with  $\text{ht } \eta = 0$  being immediate from (4.5). For the inductive step assume that we have proved the result for all  $\eta \in Q^+$  with  $\text{ht } \eta < k$ . To prove the result for  $\text{ht } \eta = k$ , it suffices to prove that for all  $j \in I$  and all monomials  $\mathbf{m}$  in  $A$  we have

$$((x_{\alpha_j}^- \otimes \mathbf{m})w) \otimes g \in \mathbf{U}(\mathfrak{g} \otimes A)(w_{\omega_i, A/\mathfrak{J}} \otimes 1),$$

where  $w \in (\mathbf{W}_A^{\omega_i} A/\mathfrak{J})_{\omega_i - \eta'}$  with  $\text{ht } \eta' = k - 1$  and  $g \in A$ . For this, we argue by a further induction on  $\deg_u \mathbf{m}$ . If  $\deg_u \mathbf{m} = 0$  then  $\mathbf{m} = \mathbf{m}_t$  and we have

$$((x_{\alpha_j}^- \otimes \mathbf{m})(w) \otimes g = (x_{\alpha_j}^- \otimes \mathbf{m})(w \otimes \mathbf{m}_t^{-1} g) \in \mathbf{U}(\mathfrak{g} \otimes A)(\mathbf{W}_A^{\omega_i} A/\mathfrak{J})_{\omega_i - \eta'}.$$

This proves that the induction on  $\deg_u \mathbf{m}$  starts. If  $\deg_u \mathbf{m} > 0$  we use (4.2) to get

$$((x_{\alpha_j}^- \otimes \mathbf{m})w) \otimes g = (x_{\alpha_j}^- \otimes \mathbf{m})(w \otimes \mathbf{m}_t^{-1} g) - \sum_q ((x_{\alpha_j}^- \otimes \mathbf{m}'_{u, q} \mathbf{m}_t)w) \otimes \mathbf{m}''_{u, q} g.$$

Both terms on the right hand side are in  $\mathbf{U}(\mathfrak{g} \otimes A)(\mathbf{W}_A^{\omega_i} A/\mathfrak{J})_{\omega_i - \eta'}$ : the first by the induction hypothesis on  $\text{ht } \eta'$  and the second by the induction hypothesis on  $\deg_u \mathbf{m}$ . This completes the proof of the surjectivity of  $\mathbf{p}$ .

To prove that  $\mathbf{p}$  is injective, recall from Section 4.1 that  $(\mathbf{W}_A^{\omega_i} A/\mathfrak{J} \otimes A)_{\mathbf{h}_{k, \ell}}$  is a free right  $A$ -module of rank equal to the dimension of  $\mathbf{W}_A^{\omega_i} A/\mathfrak{J}$ . Hence if  $K$  is the kernel of  $\mathbf{p}$  we have an isomorphism of right  $A$ -modules,

$$W_A(\omega_i) \cong K \oplus (\mathbf{W}_A^{\omega_i} A/\mathfrak{J} \otimes A)_{\mathbf{h}_{k, \ell}}.$$

Using (4.3) we see that for any maximal ideal  $\mathfrak{J}$  in  $A$ ,

$$\dim(W_A(\omega_i) \otimes_A A/\mathfrak{J}) = \dim \mathbf{W}_A^{\omega_i} A/\mathfrak{J} = \dim \mathbf{W}_A^{\omega_i} A/\mathfrak{J} = \dim((\mathbf{W}_A^{\omega_i} A/\mathfrak{J} \otimes A)_{\mathbf{h}_{k, \ell}} \otimes_A A/\mathfrak{J}).$$

Therefore,  $K \otimes_A A/\mathfrak{J} = 0$  and so by Nakayama's Lemma  $K_{\mathfrak{J}} = 0$  for all  $\mathfrak{J} \in \text{Max } A$ . Thus,  $K = 0$ .  $\square$

## 5. PROOF OF THEOREM 2 AND COROLLARY 2.2

We continue to assume that  $A = \mathcal{R}_{k,\ell}$ ,  $k, \ell \in \mathbf{Z}_+$  and that  $\mathfrak{J}$  is the maximal ideal of  $A$  generated by  $\{t_1 - 1, \dots, t_k - 1, u_1, \dots, u_\ell\}$ . We also use the comultiplication  $\mathbf{h}_{k,\ell}$  and denote it by just  $\mathbf{h}$ . Let  $\mathfrak{M}_A \subset A$  be the set of monomials in the generators  $u_r$ ,  $t_s^{\pm 1}$ ,  $1 \leq r \leq \ell$ ,  $1 \leq s \leq k$ .

**5.1.** The following proposition, together with Lemma 3.5 and Proposition 4.4, completes the proof of Theorem 2.

**Proposition.** *Suppose that  $V_s$ ,  $1 \leq s \leq M$  are  $\mathfrak{g} \otimes A$ -modules such that there exists  $N \in \mathbf{Z}_+$  with*

$$(\mathfrak{n}^+ \otimes \mathfrak{J}^N)V_s = 0, \quad 1 \leq s \leq M. \quad (5.1)$$

Then

$$((V_1 \otimes A)_{\mathbf{h}} \otimes \dots \otimes (V_M \otimes A)_{\mathbf{h}})^{\mathfrak{n}^+ \otimes A} = (V_1^{\mathfrak{n}^+ \otimes A} \otimes A)_{\mathbf{h}} \otimes \dots \otimes (V_M^{\mathfrak{n}^+ \otimes A} \otimes A)_{\mathbf{h}}.$$

**5.2.** The first step in the proof of Proposition 5.1 is the following. We need some notation. Let  $V$  be a  $\mathfrak{g} \otimes A$ -module and let  $K \in \mathbf{Z}_+$ . Define

$$V_{\geq K} = \{v \in V : (\mathfrak{n}^+ \otimes \mathbf{m})v = 0, \mathbf{m} \in \mathfrak{M}_A, |\deg_t \mathbf{m}| \geq K\}.$$

Note that  $V_{\geq 0} = V^{\mathfrak{n}^+ \otimes A}$ .

**Lemma.** *Let  $V$  be a  $\mathfrak{g} \otimes A$ -module and  $K \in \mathbf{Z}_+$ . Then*

$$((V \otimes A)_{\mathbf{h}})_{\geq K} = V_{\geq K} \otimes A. \quad (5.2)$$

In particular,

$$(V \otimes A)_{\mathbf{h}}^{\mathfrak{n}^+ \otimes A} = V^{\mathfrak{n}^+ \otimes A} \otimes A. \quad (5.3)$$

*Proof.* Let  $v_{\mathbf{h}} \in (V \otimes A)_{\mathbf{h}}$  and write,  $v_{\mathbf{h}} = \sum_p v_p \otimes g_p$ , where  $\{g_p\}_p$  is a linearly independent subset of  $A$ . By (4.2) we have

$$(x \otimes \mathbf{m})v_{\mathbf{h}} = \sum_p (x \otimes \mathbf{m})v_p \otimes \mathbf{m}_t g_p + \sum_{p,q} (x \otimes \mathbf{m}'_{u,q} \mathbf{m}_t) v_p \otimes \mathbf{m}''_{u,q} \mathbf{m}_t g_p.$$

with  $\deg_u \mathbf{m}'_{u,q} < \deg_u \mathbf{m}$ . Since  $\deg_t \mathbf{m}'_{u,q} \mathbf{m}_t = \deg_t \mathbf{m}$ , it follows that

$$V_{\geq K} \otimes A \subset ((V \otimes A)_{\mathbf{h}})_{\geq K}.$$

We prove the reverse inclusion by induction on  $\deg_u \mathbf{m}$ . Let  $v_{\mathbf{h}} \in ((V \otimes A)_{\mathbf{h}})_{\geq K}$  and let  $\deg_t \mathbf{m} \geq K$ . If  $\deg_u \mathbf{m} = 0$ , then

$$0 = (x \otimes \mathbf{m})v_{\mathbf{h}} = \sum_p (x \otimes \mathbf{m})v_p \otimes g_p \mathbf{m}.$$

Since the set  $\{g_p \mathbf{m}\}_p$  is also linearly independent, we see that  $(x \otimes \mathbf{m})v_p = 0$  for all  $p$ . If  $\deg_u \mathbf{m} > 0$ , we use (4.2) to get

$$0 = (x \otimes \mathbf{m})v_{\mathbf{h}} = \sum_p (x \otimes \mathbf{m})v_p \otimes \mathbf{m}_t g_p + \sum_{p,q} (x \otimes \mathbf{m}'_{u,q} \mathbf{m}_t) v_p \otimes \mathbf{m}''_{u,q} \mathbf{m}_t g_p.$$

Since  $\deg_u \mathbf{m}'_{u,q} < \deg_u \mathbf{m}$  all terms in the second sum are zero by the induction hypothesis, and the linear independence of the set  $\{\mathbf{m}_t g_p\}_p$  gives  $(x \otimes \mathbf{m})v_p = 0$  for all  $p$ .  $\square$

**5.3.**

**Proposition.** *Let  $U, V$  be  $\mathfrak{g} \otimes A$ -modules and suppose that for some  $N \in \mathbf{Z}_+$*

$$(\mathfrak{n}^+ \otimes \mathfrak{J}^N)V = 0.$$

*Then*

$$(U \otimes (V \otimes A)_{\mathbf{h}})^{\mathfrak{n}^+ \otimes A} = U^{\mathfrak{n}^+ \otimes A} \otimes (V^{\mathfrak{n}^+ \otimes A} \otimes A). \quad (5.4)$$

Before proving this proposition, we establish Proposition 5.1. The argument is by induction on  $M$ , with (5.3) showing that induction begins at  $M = 1$ . For  $M > 1$ , take

$$U = (V_1 \otimes A)_{\mathbf{h}} \otimes \cdots \otimes (V_{M-1} \otimes A)_{\mathbf{h}}, \quad V = V_M.$$

The induction hypothesis applies to  $U$  and together with Proposition 5.3 completes the inductive step.

**5.4.**

**Lemma.** *Let  $A = \mathcal{R}_{k,\ell}$  with  $k > 0$ . Let  $V$  be a  $\mathfrak{g} \otimes A$ -module and suppose that  $(\mathfrak{n}^+ \otimes \mathfrak{J}^N)V = 0$  for some  $N \in \mathbf{Z}_+$ . Then for all  $K \in \mathbf{Z}_+$  we have*

$$V^{\mathfrak{n}^+ \otimes A} = V_{\geq K}.$$

*Proof.* It suffices to prove that  $V_{\geq K} \subset V_{\geq K-1}$  for all  $K \geq 1$ . Since  $(1 - t_1^{\pm 1})^N \in \mathfrak{J}^N$  we have

$$0 = (x \otimes \mathbf{m}(1 - t_1^{\pm 1})^N)v = (x \otimes \mathbf{m})v + \sum_{s=1}^N (-1)^s \binom{N}{s} (x \otimes \mathbf{m}t_1^{\pm s})v, \quad (5.5)$$

for all  $x \in \mathfrak{n}^+$ ,  $\mathbf{m} \in \mathfrak{M}_A$  and  $v \in V$ . Suppose that  $v \in V_{\geq K}$  and take  $\mathbf{m} \in \mathfrak{M}_A$  with  $|\deg_t \mathbf{m}| = K-1$ . If  $\deg_t \mathbf{m} \geq 0$  (respectively,  $\deg_t \mathbf{m} < 0$ ) then  $|\deg_t \mathbf{m}t_1^{\pm s}| \geq K$  (respectively,  $|\deg_t \mathbf{m}t_1^{\pm s}| \geq K$ ) for all  $s > 0$ . Thus we conclude that all terms in the sum in (5.5) with the appropriate sign choice equal zero hence  $(x \otimes \mathbf{m})v = 0$  and so  $v \in V_{\geq K-1}$ .  $\square$

**5.5.**

**Lemma.** *Let  $A = \mathcal{R}_{0,\ell}$ . Let  $V$  be a  $\mathfrak{g} \otimes A$ -module and suppose that  $(\mathfrak{n}^+ \otimes \mathfrak{J}^N)V = 0$  for some  $N \in \mathbf{Z}_+$ . Let  $K \geq N \in \mathbf{Z}_+$ . Then*

$$V^{\mathfrak{n}^+ \otimes A} \otimes A = \{v_{\mathbf{h}} \in (V \otimes A)_{\mathbf{h}} : (\mathfrak{n}^+ \otimes \mathbf{m})v_{\mathbf{h}} = 0, \mathbf{m} \in \mathfrak{M}_A, \deg_u \mathbf{m} \geq K\} \quad (5.6)$$

*Proof.* Since  $(V \otimes A)_{\mathbf{h}}$  is a  $(\mathfrak{g} \otimes A, A)$ -bimodule the sets on both sides of (5.6) are right  $A$ -modules. Hence if  $v_{\mathbf{h}}$  is an element of the set on the right hand side of (5.6) then  $v_{\mathbf{h}}u_j^s$  is also in the right hand side of (5.6) for all  $s \in \mathbf{Z}_+$ . Write  $v_{\mathbf{h}} = \sum_p v_p \otimes g_p$ , where  $\{g_p\}_p$  is a linearly independent subset of  $A$ . Since the  $u_j$ ,  $1 \leq j \leq \ell$  are primitive and  $u_j^s \in \mathfrak{J}^N$  if  $s \geq N$ , we have for all  $0 \leq r \leq N$

$$\begin{aligned} 0 &= (x \otimes u_j^{(K+N-r)})(v_{\mathbf{h}})u_j^{(r)} = \sum_{s=0}^N \left( \sum_p ((x \otimes u_j^{(s)})v_p) \otimes u_j^{(K+N-r-s)}g_p u_j^{(r)} \right) \\ &= \sum_{s=0}^N \binom{K+N-s}{r} \left( \sum_p ((x \otimes u_j^{(s)})v_p) \otimes u_j^{(K+N-s)}g_p \right). \end{aligned}$$



We claim that the matrix  $C(N, K) = ((\binom{K+N-s}{r})_{0 \leq s, r \leq N})$  is invertible. Assuming the claim, we get

$$\sum_p ((x \otimes u_j^{(s)})v_p) \otimes u_j^{(K+N-s)} g_p = 0, \quad 0 \leq s \leq N,$$

and since the  $g_p$  are linearly independent this implies that

$$(x \otimes u_j^{(s)})v_p = 0, \quad 0 \leq s \leq N$$

and so  $(x \otimes u_j^s)v_{\mathbf{h}} = 0$  for all  $x \in \mathfrak{n}^+$ ,  $s \in \mathbf{Z}_+$ .

Now, let  $\mathbf{m} \in \mathfrak{M}_A$  and let  $\alpha \in \Phi^+$ . Then  $(h_\alpha \otimes \mathbf{m})v_{\mathbf{h}}$  is also an element of the right hand side of (5.6) and hence by the preceding argument, we get

$$\begin{aligned} 0 &= (x_\alpha \otimes 1)(h_\alpha \otimes \mathbf{m})v_{\mathbf{h}} \\ &= (h_\alpha \otimes \mathbf{m})(x_\alpha \otimes 1)v_{\mathbf{h}} - 2(x_\alpha \otimes \mathbf{m})v_{\mathbf{h}} = -(2x_\alpha \otimes \mathbf{m})v_{\mathbf{h}}, \end{aligned}$$

thus proving that  $v_{\mathbf{h}} \in (V \otimes A)_{\mathbf{h}}^{\mathfrak{n}^+ \otimes A} = V^{\mathfrak{n}^+ \otimes A} \otimes A$  by (5.3).

To prove the claim, let  $u$  be an indeterminate and let  $\{p_r \in \mathbf{C}[u] : 0 \leq r \leq N\}$  be a collection of polynomials such that  $\deg p_r = r$  (in particular, we assume that  $p_0$  is a non-zero constant polynomial). Then for any tuple  $(a_0, \dots, a_N) \in \mathbf{C}^{N+1}$ , we have  $\det(p_r(a_s))_{0 \leq r, s \leq N} = c \det(a_s^r)_{0 \leq r, s \leq N} = c \prod_{0 \leq r < s \leq N} (a_s - a_r)$ , where  $c$  is the product of highest coefficients of the  $p_r$ ,  $0 \leq r \leq N$ . Since  $\binom{u}{r}$  is a polynomial in  $u$  of degree  $r$  with highest coefficient  $1/r!$ , we obtain with  $a_s = N + K - s$ ,

$$\det C(N, K) = \left( \prod_{r=1}^N r! \right)^{-1} \prod_{0 \leq r < s \leq N} (r - s) = (-1)^{N(N+1)/2}. \quad \square$$

**5.6.** Now we have all the necessary ingredients to prove Proposition 5.3.

*Proof of Proposition 5.3.* Let  $v_{\mathbf{h}} \in (U \otimes (V \otimes A)_{\mathbf{h}})^{\mathfrak{n}^+ \otimes A}$  and write  $v_{\mathbf{h}} = \sum_{p,s} w_s \otimes v_{s,p} \otimes g_p$ , where  $\{w_s\}_s$  and  $\{g_p\}_p$  are linearly independent subsets of  $U$  and  $A$  respectively. We have

$$0 = (x \otimes \mathbf{m})v_{\mathbf{h}} = \sum_{s,p} \left( ((x \otimes \mathbf{m})w_s) \otimes v_{s,p} \otimes g_p + w_s \otimes (x \otimes \mathbf{m})(v_{s,p} \otimes g_p) \right) \quad (5.7)$$

$$\begin{aligned} &= \sum_{s,p} ((x \otimes \mathbf{m})w_s) \otimes v_{s,p} \otimes g_p \\ &\quad + \sum_{s,p} w_s \otimes \left( (x \otimes \mathbf{m})v_{s,p} \otimes g_p \mathbf{m}_t + \sum_q (x \otimes \mathbf{m}'_{u,q} \mathbf{m}_t) v_{s,p} \otimes \mathbf{m}''_{u,q} \mathbf{m}_t g_p \right), \end{aligned} \quad (5.8)$$

Suppose first that  $A = \mathcal{R}_{k,\ell}$  with  $k > 0$  and let  $K = \max_p |\deg_t g_p| + 1$ . If  $\mathbf{m}$  is such that  $|\deg_t \mathbf{m}| \geq K$ , then the set  $\{g_p\}_p$  is linearly independent from the set  $\{\mathbf{m}''_{u,q} \mathbf{m}_t g_p\}_{p,q}$  and hence we must have that

$$\sum_{s,p} ((x \otimes \mathbf{m})w_s) \otimes v_{s,p} \otimes g_p = 0, \quad \sum_{s,p} w_s \otimes (x \otimes \mathbf{m})(v_{s,p} \otimes g_p) = 0 \quad (5.9)$$

and using the linear independence of the elements  $\{w_s\}_s$  we conclude that for all  $s$

$$\sum_p (v_{s,p} \otimes g_p) \in ((V \otimes A)_{\mathfrak{h}})_{\geq K} = V_{\geq K} \otimes A = V^{\mathfrak{n}^+ \otimes A} \otimes A,$$

using (5.2) and Lemma 5.4. This proves Proposition 5.3 in the case when  $k > 0$ .

Suppose now that  $A = \mathcal{R}_{0,\ell}$  and let  $N \in \mathbf{Z}_+$  be such that  $(\mathfrak{n}^+ \otimes \mathfrak{J}^N)V = 0$ . Let  $K = N + 1 + \max_p \{\deg_u g_p\}$  and let  $x \in \mathfrak{n}^+$ . If  $\deg_u \mathbf{m} \geq K$  then  $\mathbf{m} \in \mathfrak{J}^N$  and so  $(x \otimes \mathbf{m})v_{s,p} = 0$ . Furthermore,  $(x \otimes \mathbf{m}'_{u,q})v_{s,p} \neq 0$  implies that  $\deg_u \mathbf{m}'_{u,q} < N$ . By Lemma 4.2 it follows that  $\deg_u \mathbf{m}''_{u,q} > \max_p \{\deg g_p\}$ . Therefore, the non-zero terms, if any, in the second sum in (5.8) are linearly independent from those in the first sum and we obtain (5.9). Furthermore, we have

$$0 = \sum_p \sum_{\{q : \deg_u \mathbf{m}'_{u,q} < N\}} (x \otimes \mathbf{m}'_{u,q})v_{s,p} \otimes \mathbf{m}''_{u,q}g_p$$

and as before we conclude that  $(x \otimes \mathbf{m}'_{u,q})v_{s,p} = 0$  when  $\deg_u \mathbf{m}'_{u,q} < N$ . Thus,  $(x \otimes \mathbf{m})(\sum_p v_{s,p} \otimes g_p) = 0$  for all  $s$  and it remains to apply Lemma 5.5.  $\square$

**5.7.** We conclude this section with a proof of Corollary 2.2. The following is a special case of Theorem 7.1 and Proposition 7.7 of [4].

**Theorem 4.** Let  $\mathfrak{J} \in \text{Max } A$ , where  $A = \mathcal{R}_{k,\ell}$ . Then

$$\dim(\mathbf{W}_A^{\omega_i} A / \mathfrak{J})^{\mathfrak{n}^+ \otimes A} = \dim(\mathbf{W}_A^{\omega_i} A / \mathfrak{J})_{\omega_i} = 1, \quad i \in I_0,$$

If  $\mathfrak{g}$  is of type  $B_n$  or  $D_n$  and  $i \notin I_0$  then

$$\dim(\mathbf{W}_A^{\omega_i} A / \mathfrak{J})_{\mu}^{\mathfrak{n}^+ \otimes A} = \begin{cases} 0, & \mu \neq \omega_{i-2j}, \\ \binom{j+k-1}{j}, & \mu = \omega_{i-2j}, i-2j \geq 0 \end{cases}$$

where  $\omega_0 = 0$ .  $\square$

Another way to formulate this result is the following. The subspace  $(\mathbf{W}_A^{\omega_i} A / \mathfrak{J})^{\mathfrak{n}^+ \otimes A}$  is an  $\mathfrak{h}$ -module with character given by

$$\text{ch}(\mathbf{W}_A^{\omega_i} A / \mathfrak{J})^{\mathfrak{n}^+ \otimes A} = \sum_{j: i-2j \geq 0} \binom{j+k-1}{j} e(\omega_{i-2j}).$$

Hence, using Theorem 2, we get

$$\text{ch} \mathbf{W}_A(\mathbf{s})^{\mathfrak{n}^+ \otimes A} = \prod_{i \in I} (\text{ch}(\mathbf{W}_A^{\omega_i} A / \mathfrak{J})^{\mathfrak{n}^+ \otimes A})^{s_i},$$

and Corollary 2.2 follows.

## 6. PROOF OF THEOREM 3

Given  $\mathbf{s} \in \mathbf{Z}_+^I$ , define  $\mu_{\mathbf{s}} \in P^+$  by  $\mu_{\mathbf{s}} = \sum_{i \in I} s_i \omega_i$  and let  $\tau_{\mathbf{s}} : W_A(\mu_{\mathbf{s}}) \rightarrow W_A(\mathbf{s})$  be the natural map of  $(\mathfrak{g} \otimes A, A)$ -bimodules defined in the above lemma and satisfying  $\tau_{\mathbf{s}}(w_{\mu_{\mathbf{s}}}) = w_{\mathbf{s}}$ . Since  $W_A(\mathbf{s})_{\mu_{\mathbf{s}}} = w_{\mathbf{s}} \otimes \mathbf{A}_{\mathbf{s}}$ , we see that any non-zero element  $\tau$  of  $\text{Hom}_{\mathfrak{g} \otimes A}(W_A(\mu_{\mathbf{s}}), W_A(\mathbf{s}))$  is given by composing  $\tau_{\mathbf{s}}$  with right multiplication by an element of  $\mathbf{A}_{\mathbf{s}}$ , i.e.  $\tau = \tau_{\mathbf{s}} \mathbf{a}$  with  $\mathbf{a} \in \mathbf{A}_{\mathbf{s}}$ .

### 6.1.

**Lemma.** Assume that  $\lambda \in P^+$  and  $\mathbf{s} \in \mathbf{Z}_+^I$  satisfy

- (i) any non-zero element of  $\text{Hom}_{\mathfrak{g} \otimes A}(W_A(\lambda), W_A(\mathbf{s}))$  is injective,
- (ii) the map  $\tau_{\mathbf{s}} : W_A(\mu_{\mathbf{s}}) \rightarrow W_A(\mathbf{s})$  is injective.

Then any non-zero element of  $\text{Hom}_{\mathfrak{g} \otimes A}(W_A(\lambda), W_A(\mu_{\mathbf{s}}))$  is injective. Moreover, if  $s_i = 0$  for all  $i \notin I_0$ , then

$$\text{Hom}_{\mathfrak{g} \otimes A}(W_A(\lambda), W_A(\mu_{\mathbf{s}})) = 0, \quad \lambda \neq \mu_{\mathbf{s}}.$$

*Proof.* Let  $\eta \in \text{Hom}_{\mathfrak{g} \otimes A}(W_A(\lambda), W_A(\mu_{\mathbf{s}}))$ . If  $\eta \neq 0$ , then  $\tau_{\mathbf{s}} \cdot \eta \in \text{Hom}_{\mathfrak{g} \otimes A}(W_A(\lambda), W_A(\mathbf{s}))$  is non-zero since  $\tau_{\mathbf{s}}$  is injective. Hence  $\tau_{\mathbf{s}} \cdot \eta$  is injective which forces  $\eta$  to be injective. If we now assume that  $\lambda \neq \mu_{\mathbf{s}}$  and that  $s_i = 0$  if  $i \notin I_0$ , then it follows from Corollary 2.2 that  $\text{Hom}_{\mathfrak{g} \otimes A}(W_A(\lambda), W_A(\mathbf{s})) = 0$  and hence it follows that  $\eta = 0$  in this case.  $\square$

**Remark.** Using Theorem 4, we see that if  $\mathfrak{g}$  is of type  $B_n$  or  $D_n$  with  $n \geq 6$  and  $i = 4$ , then  $(\mathbf{W}_A^{\omega_4} A / \mathfrak{J})_{\omega_2}^{n^+ \otimes A} \neq 0$  or equivalently

$$\text{Hom}_{\mathfrak{g} \otimes A}(\mathbf{W}_A^{\omega_2} A / \mathfrak{J}, \mathbf{W}_A^{\omega_4} A / \mathfrak{J}) \neq 0.$$

Using Proposition 4.4 and (4.1) we get

$$\text{Hom}_{\mathfrak{g} \otimes A}(W_A(\omega_2), W_A(\omega_4)) \neq 0,$$

which in particular proves that the last assertion of the above Lemma and hence Theorem 3 fail in this case.

**6.2.** From now on, we shall assume that  $\mathbf{s} \in \mathbf{Z}_+^I$  is such that  $s_i = 0$  if  $i \notin I_0$ . By Corollary 2.2, we see that  $\text{Hom}_{\mathfrak{g} \otimes A}(W_A(\lambda), W_A(\mu_{\mathbf{s}})) = 0$  if  $\lambda \neq \mu_{\mathbf{s}}$  and the first condition of Lemma 6.1 is trivially satisfied. Hence Theorem 3 will follow if we show that  $\mu_{\mathbf{s}}$  satisfies both conditions in Lemma 6.1. By the discussion at the start of Section 5, we see that proving that  $\mu_{\mathbf{s}}$  satisfies the first condition is equivalent to proving that  $\tau_{\mathbf{s}} \mathbf{a}$  is injective for all  $\mathbf{a} \in \mathbf{A}_{\mathbf{s}}$ . In other words, Theorem 3 follows if we establish the following.

**Proposition.** Let  $\mathbf{s} \in \mathbf{Z}_+^I$  be such that  $s_i = 0$  if  $i \notin I_0$ . For all  $\mathbf{a} \in \mathbf{A}_{\mathbf{s}}$ , the canonical map  $\tau_{\mu_{\mathbf{s}}} \mathbf{a} : W_A(\mu_{\mathbf{s}}) \rightarrow W_A(\mathbf{s})$  given by extending  $w_{\mu_{\mathbf{s}}} \rightarrow w_{\mathbf{s}} \mathbf{a}$  is injective, in the following cases:

- (i)  $A = \mathcal{R}_{0,1}$  or  $\mathcal{R}_{1,0}$ ,
- (ii)  $A = \mathcal{R}_{k,\ell}$ ,  $\mathfrak{g} = \mathfrak{sl}_{n+1}$  and  $\mathbf{s} = (s, 0, \dots, 0) \in \mathbf{Z}_+^I$ ,  $s > 0$ .

The rest of the section is devoted to proving the proposition.

**6.3.** We begin by proving the following Lemma.

**Lemma.** Let  $A$  be a finitely generated integral domain. Let  $\mathbf{s} \in \mathbf{Z}_+^I$  be such that  $s_i = 0$  if  $i \notin I_0$ . Then  $\tau_{\mu_{\mathbf{s}}} \mathbf{a} : W_A(\mu_{\mathbf{s}}) \rightarrow W_A(\mathbf{s})$  is injective for  $\mathbf{a} \in \mathbf{A}_{\mathbf{s}} \setminus \{0\}$  if and only if  $\tau_{\mu_{\mathbf{s}}}$  is injective.

*Proof.* Consider the map  $\rho_{\mathbf{a}} : W_A(\mathbf{s}) \rightarrow W_A(\mathbf{s})$  given by

$$\rho_{\mathbf{a}}(w) = w\mathbf{a}, \quad w \in W_A(\mathbf{s}).$$

This is clearly a map of  $(\mathfrak{g} \otimes A, \mathbf{A}_s)$ -bimodules. Since

$$W_A(\mathbf{s})_{\mu_s} = w_s \otimes \mathbf{A}_s,$$

and  $\mathbf{A}_s$  is an integral domain, it follows that the restriction of  $\rho_a$  to  $W_A(\mathbf{s})_{\mu_s}$  is injective, and so

$$\ker \rho_a \cap W_A(\mathbf{s})_{\mu_s} = \{0\}.$$

Since  $\text{wt } W_A(\mathbf{s}) \subset \mu_s - Q^+$ , it follows that if  $\ker \rho_a$  is non-zero, there must exist  $w' \in \ker \rho_a$  with

$$(\mathfrak{n}^+ \otimes A)w' = 0.$$

But this is impossible by Corollary 2.2 and hence  $\ker \rho_a = 0$ . Since  $\tau_{\mu_s} \mathbf{a} = \rho_a \tau_{\mu_s}$ , the Lemma follows.  $\square$

**6.4.** We now prove that  $\tau_{\mu_s}$  is injective. This was proved in [6] for  $\mathfrak{g} = \mathfrak{sl}_2$  and  $A = \mathcal{R}_{1,0}$  and in [7] for  $\mathfrak{g} = \mathfrak{sl}_{n+1}$ ,  $\mathbf{s} = (s, 0, \dots, 0) \in \mathbf{Z}_+^n$ ,  $s > 0$  and for any finitely generated integral domain  $A$ .

Since

$$\tau_s W_A(\mu_s)_{\mu_s} \cong_{\mathbf{A}_{\mu_s}} (\mathbf{U}(\mathfrak{g} \otimes A)w_s)_{\mu_s} \cong_{\mathbf{A}_{\mu_s}} \mathbf{A}_{\mu_s},$$

the following proposition completes the proof of Proposition 6.2.

**Proposition.** *Let  $\mu \in P^+$  and let  $\pi : W_A(\mu) \rightarrow W$  be a surjective map of  $(\mathfrak{g} \otimes A, \mathbf{A}_\mu)$ -bimodules such that the restriction of  $\pi$  to  $W_A(\mu)_\mu$  is an isomorphism of right  $\mathbf{A}_\mu$ -modules. If  $A = \mathcal{R}_{0,1}$  or  $\mathcal{R}_{1,0}$  and  $\mu = \sum_{i \in I_0} s_i \omega_i$ , then  $\pi$  is an isomorphism.*

**6.5.** Assume from now on that  $A$  is either  $\mathcal{R}_{0,1}$  or  $\mathcal{R}_{1,0}$ . The following is well-known.

**Proposition.** *For all  $r \in \mathbf{Z}_+$ , the ring  $(\mathcal{R}_{0,1}^{\otimes r})^{S_r}$  is isomorphic to  $\mathcal{R}_{0,r}$  and  $(\mathcal{R}_{1,0}^{\otimes r})^{S_r}$  is isomorphic to  $\mathbf{C}[t_1, t_2, \dots, t_r, t_r^{-1}]$ .*

The proposition implies that  $\text{Max } \mathbf{A}_\lambda$  is an irreducible variety. Given  $\lambda \in P^+$ , define  $\mathcal{D}_\lambda \subset \text{Max } \mathbf{A}_\lambda$  by:  $\mathbf{I} \in \mathcal{D}_\lambda$  if and only if the  $S_{r_\lambda}$ -orbit of  $\text{sym}_\lambda \mathbf{I}$  is of maximal size, i.e.,  $\text{sym}_\lambda \mathbf{I}$  is the  $S_{r_\lambda}$ -orbit of  $((t - a_{1,1}), \dots, (t - a_{1,r_1}), \dots, (t - a_{n,1}), \dots, (t - a_{n,r_n})) \in (\text{Max } A)^{\times r_\lambda}$  for some  $a_{i,r} \in \mathbf{C}$  (respectively  $a_{i,r} \in \mathbf{C}^\times$ ) with  $a_{i,r} \neq a_{j,s}$  if  $(i, r) \neq (j, s)$ . The set of such orbits is clearly Zariski open in  $\text{Max } \mathbb{A}_\lambda$ . Since  $\text{sym}_\lambda$  induces an isomorphism of algebraic varieties  $\text{Max } \mathbb{A}_\lambda \rightarrow \text{Max } \mathbf{A}_\lambda$ , we conclude that  $\mathcal{D}_\lambda$  is Zariski open, hence is dense in  $\text{Max } \mathbf{A}_\lambda$ . Therefore, given any non-zero  $\mathbf{a} \in \mathbf{A}_\lambda$  there exists  $\mathbf{I} \in \mathcal{D}_\lambda$  with  $\mathbf{a} \notin \mathbf{I}$ .

**6.6.** We shall need the following theorem.

**Theorem 5.** Let  $A = \mathcal{R}_{0,1}$  or  $\mathcal{R}_{1,0}$  and let  $\lambda = \sum_{i \in I} r_i \omega_i \in P^+$ .

(i) The right  $\mathbf{A}_\lambda$ -module  $W_A(\lambda)$  is free of rank  $d_\lambda$ , where

$$d_\lambda = \prod_{i \in I} (\dim \mathbf{W}_A^{\omega_i}(A/\mathfrak{J}))^{r_i},$$

for any  $\mathfrak{J} \in \text{Max } A$ .

(ii) Let  $\mathbf{I} \in \mathcal{D}_\lambda$ . Then

$$\mathbf{W}_A^\lambda(\mathbf{A}_\lambda/\mathbf{I}) \cong \bigotimes_{i \in I} \bigotimes_{r=1}^{r_i} \mathbf{W}_A^{\omega_i}(A/\mathfrak{I}_{i,r}),$$

where  $\mathfrak{I}_{i,r} \in \text{Max } A$  is the ideal generated by  $(t - a_{i,r})$ . If, in addition, we have  $r_i = 0$  for  $i \notin I_0$ , then  $\mathbf{W}_A^\lambda(\mathbf{A}_\lambda/\mathbf{I})$  is an irreducible  $\mathfrak{g} \otimes A$ -module.

Part (i) of the Theorem was proved in [6] for  $\mathfrak{sl}_2$ , in [5] for  $\mathfrak{sl}_{r+1}$  and in [8] for algebras of type  $A, D, E$ . The general case can be deduced from the quantum case, using results of [1, 10, 14]. Part (ii) of the Theorem was proved in [6] in a different language and in [4] in the language of this paper.

### 6.7.

*Proof of Proposition 6.4.* Let  $\{w_s\}_{1 \leq s \leq d_\mu}$  be an  $\mathbf{A}_\mu$ -basis of  $W_A(\mu)$  (cf. Theorem 5(i)). Then for all  $\mathbf{I} \in \text{max } \mathbf{A}_\mu$ ,  $\{w_s \otimes 1\}_{1 \leq s \leq d_\mu}$  is a  $\mathbf{C}$ -basis of  $\mathbf{W}_A^\mu \mathbf{A}_\mu/\mathbf{I}$ . Suppose that  $w \in \ker \pi$  and write

$$w = \sum_{s=1}^{d_\mu} w_s \mathbf{a}_s, \quad \mathbf{a}_s \in \mathbf{A}_\mu.$$

If  $w \neq 0$ , let  $\mathbf{a}$  be the product of the non-zero elements of the set  $\{\mathbf{a}_s : 1 \leq s \leq d_\mu\}$ . Since  $\mathbf{A}_\mu$  is an integral domain we see that  $\mathbf{a} \neq 0$ . By the discussion in Section 6.5 we can choose  $\mathbf{I} \in \mathcal{D}_\mu$  with  $\mathbf{a} \notin \mathbf{I}$ . Then  $\mathbf{a}_s \neq 0$  implies that  $\mathbf{a}_s \notin \mathbf{I}$  and hence  $\bar{w} := w \otimes 1 = \sum_{s=1}^{d_\mu} w_s \otimes \bar{\mathbf{a}}_s \neq 0$ , where  $\bar{\mathbf{a}}_s$  is the canonical image of  $\mathbf{a}_s$  in  $\mathbf{A}_\mu/\mathbf{I}$ . Notice that Theorem 5(ii) implies that  $\mathbf{W}_A^\mu(\mathbf{A}_\mu/\mathbf{I})$  is a simple  $\mathfrak{g} \otimes A$ -module.

Since  $\pi$  is surjective,  $W$  is generated by  $\pi(w_\mu)$ . Setting  $W' = \pi(W_A(\mu)\mathbf{I})$ , we see that

$$W'_\mu = \pi((W_A(\mu)\mathbf{I})_\mu) = \pi(w_\mu)\mathbf{I}.$$

In particular, this proves that  $\pi(w_\mu) \notin W'$ , hence  $W'$  is a proper submodule of  $W$  and

$$(W/W')\mathbf{I} = 0.$$

This implies that  $\pi$  induces a well-defined non-zero surjective homomorphism of  $\mathfrak{g} \otimes A$ -modules  $\bar{\pi} : \mathbf{W}_A^\mu(\mathbf{A}_\mu/\mathbf{I}) \rightarrow W/W' \rightarrow 0$ . In fact since  $\mathbf{W}_A^\mu(\mathbf{A}_\mu/\mathbf{I})$  is simple, we see that  $\bar{\pi}$  is an isomorphism. But now we have

$$0 = \pi(w) = \bar{\pi}(\bar{w}),$$

forcing  $\bar{w} = 0$  which is a contradiction caused by our assumption that  $w \neq 0$ .

The proof of Proposition 6.4 is complete.  $\square$

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